T distribution

Monday, May 27, 2024 11:22

$X \sim N(\mu, \sigma^2)$

$U = \frac{X-\mu}{\sigma} \sim N(0,1)$

$t = \frac{\overset{\to N(0,1)}{U}}{\sqrt{\underset{\to \chi^2(p)}{V}/p}}$. $W = V$.

$t = \frac{U}{\sqrt{V/p}}$ $U \sim N(0,1)$
$V \sim \chi^2(p)$, U and V indep.
$\hookrightarrow \text{Gamma}(\frac{p}{2}, 2)$
$\alpha = \frac{p}{2}\ \beta = 2$

$$f_{U,V}(u,v) = f_U(u) f_V(v)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}u^2\} \cdot \frac{1}{\Gamma(\frac{p}{2}) 2^{p/2}} v^{(\frac{p}{2})-1} e^{-\frac{v}{2}}$$

$t = \frac{u}{\sqrt{v/p}}$, $w = v$

思路：知道 U.V. 以及他们的 PDF 求 $t = \frac{u}{\sqrt{v/p}}$ PDF
本质上就是用 change of variable.

$U = t\, w^{\frac{1}{2}} \cdot p^{-\frac{1}{2}}$ $\frac{du}{dw} = p^{-\frac{1}{2}} t \cdot \frac{1}{2} w^{-\frac{1}{2}}$ $\frac{du}{dt} = \sqrt{\frac{w}{p}}$

$V = W$ $\frac{dv}{dw} = 1$ $\frac{dw}{dt} = 0$

$$\therefore |J| = \left|-\sqrt{\frac{w}{p}}\right| = \sqrt{\frac{w}{p}}$$

$$f_{T,W}(t,w) = f_U(u) f_V(v) \cdot |J|$$

$$= \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2} t^2 \frac{w}{p}\} \cdot \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \cdot w^{\frac{p}{2}-1} e^{-\frac{w}{2}} \ |J|$$

求 T 的 marginal PDF

$$f_T(t) = \int_{-\infty}^{+\infty} f_{T,W}(t,w) \sqrt{\frac{w}{p}}\, dw$$

这里需要合并.

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \int \exp\{-\tfrac{1}{2} \frac{t^2}{p} w\} \cdot w^{\frac{p}{2}-1} \cdot e^{-\frac{w}{2}} \frac{w^{\frac{1}{2}}}{\sqrt{p}} dw.$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}}} \int \exp\{-\tfrac{1}{2}(\frac{t^2}{p}+1) \cdot w\}\ w^{\frac{p}{2}-1+\frac{1}{2}} \cdot p^{-\frac{1}{2}} dw$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}} p^{\frac{1}{2}}} \int \exp\{-\tfrac{1}{2}(\frac{t^2}{p}+1) w\}\ w^{\frac{p}{2}-\frac{1}{2}}\ dw$$

好呀，这是个 Gamma 的 kernel
$\exp\{-x/\beta\} \cdot x^{\alpha-1}$

$$\exp\{-w / \underbrace{[\tfrac{1}{2}(\tfrac{t^2}{p}+1)]^{-1}}\} \cdot w^{\frac{p}{2}-\frac{1}{2}-1+1}$$

$$w^{\overbrace{(\frac{p}{2}+\frac{1}{2})}^{}-1}$$

$\underline{\beta} = \left(\frac{\frac{t^2}{p}+1}{2}\right)^{-1}$ $\underline{\alpha}$

$$= \Gamma(\alpha) \cdot \beta^{\alpha} = \Gamma(\tfrac{p}{2}+\tfrac{1}{2}) \cdot \left(\frac{2}{\frac{t^2}{p}+1}\right)^{\frac{p+1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\frac{p}{2}) 2^{\frac{p}{2}} p^{\frac{1}{2}}}\ \Gamma(\tfrac{p}{2}+\tfrac{1}{2}) \cdot \left(\frac{2}{\frac{t}{p}+1}\right)^{\frac{p+1}{2}}$$

$$= \frac{1}{(p\pi)^{\frac{1}{2}}} \cdot \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{1}{2^{\cancel{\frac{p+1}{2}}}} \cdot (\cancel{2})^{\frac{p+1}{2}} \cdot \frac{1}{(\frac{t}{p}+1)^{\frac{p+1}{2}}}$$

$$= \frac{1}{(p\pi)^{\frac{1}{2}}} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \cdot \frac{1}{(\frac{t}{p}+1)^{\frac{p+1}{2}}}$$